Calculation of sound fields in the ocean by the parabolic equation method

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It is shown that the parabolic equation method is an operator analog of the Wentzel–Kramers–Brillouin (WKB) method; the foundation is given for its application to the solution of cylindrically symmetrical problems for the three-dimensional Helmholtz equation. A numerical example is given.

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With a view toward the application of efficient numerical methods,[1] we adopt an acoustic model of the ocean in the form of a liquid layer bounded by two plane-parallel free boundaries. For the suppression of reflections from the lower boundary we place a strongly absorbing layer near it. For this model the sound pressure field p of a point harmonic volume-velocity source is the solution of the following boundary-value problem for the Helmholtz equation[2]:

$$\begin{gathered}\Delta u+K^2(x,y,z)u=4\pi\delta(x,y,z-z_1),\\ u(x,y,+0)=0,\quad u(x,y,H-0)=0,\\ \operatorname{Im}K^2>0\Rightarrow \lim_{x^2+y^2+z^2\to\infty}u=0,\quad p=u\sqrt{\rho},\end{gathered}\tag{1}$$

where $\rho=\rho(x, y, z)$ is the complex density of the medium,

$$K^2=\frac{\omega^2}{c^2(x,y,z)}+\frac{1}{2\rho}\Delta\rho-\frac{3}{4}\left(\frac{1}{\rho}\nabla\rho\right)^2,$$

$c = c(x, y, z)$ is the complex sound velocity in the medium, ω is the cyclic frequency, H is the thickness of the layer, the origin of the Cartesian coordinate system (x, y, z) is situated on the upper boundary, the depth axis Oz is directed toward the lower boundary and is perpendicular to it, and z_1 is the depth of the source.

The parabolic equation method, first proposed by Leontovich and Fok,[3] is initially applied to the example of the boundary-value problem for the simpler two-dimensional Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}+K^2(z)u=0\tag{2}$$

in a strip $0 < z < H$ with free boundaries. For Eq. (2) the approach of Leontovich and Fok entails the separation of a rapidly varying factor $\exp(iK_0x)$ in the solution, i.e., making the substitution for the unknown function $u = \exp(iK_0x)v$, and then, in the resulting differential equation for v, neglecting the term $\partial^2 v/\partial x^2$, which is comparatively small for an appropriate choice of the average wave number K_0; as a result, we obtain the approximate equation for u

$$\frac{\partial v}{\partial x}=iK_0v+\frac{i}{2K_0}\left(\frac{\partial^2 v}{\partial z^2}+(K^2(z)-K_0^2)v\right).\tag{3}$$

Previously,[4,5] a factorization technique has been used for the derivation of Eq. (3). We now describe an improved version of that technique.

Following Tappert,[5] we define the transverse differential operator T by the relation $T=\partial^2/\partial z^2+K^2(z)$. The domain D(T) of T is a set of sufficiently smooth functions v(z) satisfying the boundary conditions $v(+0) = 0$ and $v(H-0) = 0$. Using the operator T, we rewrite problem (2) in the form

$$\frac{\partial^2 u}{\partial x^2}+Tu=0.\tag{4}$$

Now u is an element of D(T). If we regard T as the operator wave number squared, Eq. (4) will have the same form as the one-dimensional Helmholtz equation. Something remarkable happens here: Any solution of Eq. (4) can also be written in the same notational form as the general solution of the one-dimensional Helmholtz equation,[6] but specifically in the form of the sum of two operator plane waves, one traveling to the right and the other to the left:

$$u(x)=\exp(ix\sqrt{T})v_+ +\exp(-ix\sqrt{T})v_-.\tag{5}$$

Here v_+ and v_- are certain elements of D(T). An operator plane wave is a set of normal modes traveling in one direction; for example, by the definition of a function of an operator

$$e^{ix\sqrt{T}}v_+=(\psi_1(z),\ldots,\psi_i(z),\ldots)\begin{pmatrix}e^{ix\sqrt{\lambda_1}} & & 0\\ \cdots & & \\ \cdots & e^{ix\sqrt{\lambda_i}} & \\ 0 & & \end{pmatrix}\begin{pmatrix}a_1\\ \cdots\\ a_i\\ \cdots\end{pmatrix}=\sum_i a_i\psi_i(z)\,e^{ix\sqrt{\lambda_i}}.$$

Here λ_i and $\psi_i(z)$ are the eigenvalues and eigenfunctions of the operator T, and a_i are the coefficients of the expansion of the function v_+ with respect to the basis of $\psi_i(z)$. The operator plane wave obeys the generalized parabolic equation

$$\frac{\partial u_\pm}{\partial x}=\pm i\sqrt{T}\,u_\pm,\tag{6}$$

in which $u_\pm=\exp(\pm ix\sqrt{T})v_\pm$.

The first-order equation (6) exactly describes the propagation of waves traveling in one direction, as in the case of the analogous one-dimensional equation. The Leontovich–Fok parabolic equation (3) is obtained from (6) by expanding $\sqrt{T}$ into a Taylor series and retaining up

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to and including first-order terms[5]:

$$\sqrt{\frac{\partial^2}{\partial z^2}+K^2(z)}=K_0\sqrt{1+\frac{K^2(z)-K_0^2}{K_0^2}+\frac{1}{K_0^2}\frac{\partial^2}{\partial z^2}}$$
$$\approx K_0+\frac{K^2(z)-K_0^2}{2K_0}+\frac{1}{2K_0}\frac{\partial^2}{\partial z^2}.$$

This approximation is satisfied only in a small interval of grazing angles and for a slightly varying function $K^2(z)$.

The error associated with this expansion can be judged by estimating the difference between the results of application of the exact and approximate operators to an eigenfunction of the operator T:

$$|\delta_i|=\left|\left(\sqrt{T}-\frac{K_0}{2}-\frac{1}{2K_0}T\right)\psi_i\right|=\left|\sqrt{\lambda_i}-\frac{K_0}{2}-\frac{\lambda_i}{2K_0}\right|\,|\psi_i|.$$

If $K_0 = \sqrt{\lambda_i}$, then, as expected on the basis of (6), $\delta_i = 0$. Otherwise, putting $\sqrt{\lambda_i} = K_0\cos\vartheta_i$, for small ϑ_i we obtain

$$|\delta_i|\approx\frac{K_0}{16}\vartheta_i^4$$

and, for example, an incorrect phase shift of π (in comparison with the exact solution) accrues at distances of the order of 360 km at a frequency of 50 Hz and angles $\vartheta \sim 10°$. Following is another expression for the field of a point source $\delta(x, t)$ (Ref. 6):

$$u(x)=\frac{\exp(i|x|\sqrt{T})}{\sqrt{T}}\delta(z-z_1). \tag{7}$$

We now consider the two-dimensional Helmholtz equation with wave number depending on both coordinates, $K = K(x, z)$, and with the same boundary conditions as before. The operator T now depends on x: $T(x) = \partial^2/\partial z^2 + K^2(x, z)$, and the corresponding operator equation takes the form

$$\frac{\partial^2 u}{\partial x^2}+T(x)u=0. \tag{8}$$

If the variables in (8) are separable, its solution can be written in the form

$$u(x)=R(T,x)v_+ + L(T,x)v_-,$$

where R and L are functions forming a fundamental system of solutions of the longitudinal differential equation. If the variables are not separable, then, as in the one-dimensional case, it is necessary to resort to approximative methods. If T varies smoothly with x, a solution can be sought in the form of two approximately plane waves, i.e., we can proceed as in the one-dimensional WKB method.[7] The solution is written in the form of two approximately plane waves by the following substitution for the unknown function:

$$\begin{pmatrix} u \\ \dfrac{\partial u}{\partial x}\end{pmatrix}=\begin{pmatrix} P^{-1} & P^{-1} \\ P & -P\end{pmatrix}\begin{pmatrix} v_+ \\ v_-\end{pmatrix},\quad P=(-T(x))^{\frac{1}{4}}.$$

Transforming to the new unknown functions in (8), we obtain

$$\frac{\partial}{\partial x}\begin{pmatrix} v_+ \\ v_-\end{pmatrix}=\left[\begin{pmatrix} P^2 & 0 \\ 0 & -P^2\end{pmatrix}+\frac{1}{2}\left(P\frac{\partial P^{-1}}{\partial x}+P^{-1}\frac{\partial P}{\partial x}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\right.$$
$$\left.+\frac{1}{2}\left(P\frac{\partial P^{-1}}{\partial x}-P^{-1}\frac{\partial P}{\partial x}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right]\begin{pmatrix} v_+ \\ v_-\end{pmatrix}. \tag{9}$$

Neglecting integration between the right- and left-traveling waves, i.e., off-diagonal terms in (9), we obtain the following equations in v_+ and v_-:

$$\frac{\partial v_+}{\partial x}=\left[iT^{1/2}+\frac{1}{2}T^{1/4}\frac{\partial T^{-1/4}}{\partial x}+\frac{1}{2}T^{-1/4}\frac{\partial T^{1/4}}{\partial x}\right]v,$$
$$\frac{\partial v_-}{\partial x}=\left[-iT^{1/2}+\frac{1}{2}T^{1/4}\frac{\partial T^{-1/4}}{\partial x}+\frac{1}{2}T^{-1/4}\frac{\partial T^{1/4}}{\partial x}\right]v_-,$$
$$T=T(x). \tag{10}$$

The generalized parabolic equations (10) approximately describe the propagation of the right- and left-traveling waves with regard for interaction between waves traveling in one direction.

If the point source is not situated near a turning point, i.e., where the relative error of terms rejected from (9) is large, an expression similar to (7) can be derived.

Next we consider cylindrically symmetrical problems for the three-dimensional Helmholtz equation. In cylindrical coordinates the Helmholtz equation (1) acquires the form

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}+K^2(r,z)u=4\pi\delta(z-z_1)\frac{\delta(r)}{r}. \tag{11}$$

In a layered medium the solution of problem (1) can be written in the form

$$u(r)=\pi i H_0^{(1)}(r\sqrt{T})\delta(z-z_1), \tag{12}$$

where $T = \partial^2/\partial z^2 + K^2(z)$ and $H_0^{(1)}$ is the Hankel function.

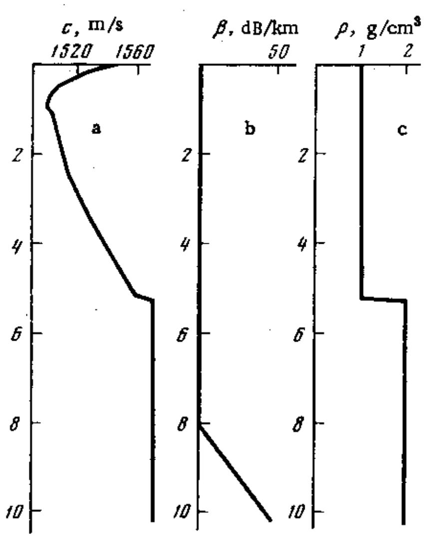

FIG. 1. Depth profiles of: a) sound velocity; b) absorption per kilometer; c) density of the medium.

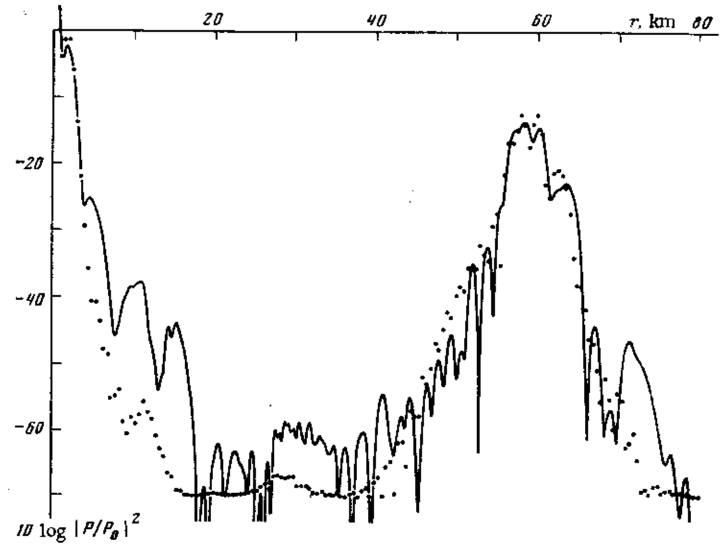

FIG. 2. Distance profile of the field of a point source, calculated according to (16) by the numerical scheme of Ref. 1 (solid curve) and by the summation of normal modes[8] (dots).

The solution (12) has the asymptotic representation with respect to Kr

$$u(r)=\pi i \exp(ir\sqrt{T})\sqrt{2/i\pi r\sqrt{T}}\,\delta(z-z_1). \tag{13}$$

In a two-dimensionally inhomogeneous medium we transform Eq. (11) to (8) by the change of variable r = exp (x) and, applying the asymptotic factorization (9), (10), after reversion to the old variables we obtain equations asymptotically describing the field of a wave diverging from the source:

$$u_+(r)=r^{-1/2}T^{-1/4}v_+(r),\quad T=T(r),$$
$$\frac{\partial v_+}{\partial r}=\left[iT^{1/2}+\frac{1}{2}T^{1/4}\frac{\partial T^{-1/4}}{\partial r}+\frac{1}{2}T^{-1/4}\frac{\partial T^{1/4}}{\partial r}\right]v_+ \tag{14}$$

where $T(r) = \partial^2/\partial z^2 + K^2(r, z)$. The source, however, is situated at a turning point of Eq. (11). To determine the initial conditions for (14) we assume that the medium near the source is layered up to distances at which the asymptotic representation (13) is valid and, neglecting the converging wave since $K^2(r, z)$ depends smoothly on r, we obtain

$$v_+(0)=(2\pi i)^{1/2}\delta(z-z_1) \tag{15}$$

because the asymptotic representation (13) satisfies Eqs. (14).

Elementary Taylor approximations of the operators in (14), (15) yield the equations

$$u_+(r,z)=\frac{1}{\sqrt{K_0 r}}\left(1-\frac{K^2(r,z)-K_0^2}{4K_0^2}-\frac{1}{4K_0^2}\frac{\partial^2}{\partial z^2}\right)v_+(r,z),$$
$$\frac{\partial v_+}{\partial r}=i\left(K_0+\frac{K^2(r,z)-K_0^2}{2K_0}+\frac{1}{2K_0}\frac{\partial^2}{\partial z^2}\right)v_+,$$
$$v_+(0,z)=\sqrt{2\pi i}\,\delta(z-z_1). \tag{16}$$

The numerical calculations according to (16) contain four types of errors: 1) asymptotic factorization errors; 2) errors of approximation of the operators; 3) errors of approximation of the generalized function $\delta(z - z_1)$; 4) errors of the numerical scheme used to solve Eqs. (16). The resultant error can be judged on the basis of the following sample calculation. Figure 1 shows the depth profile of the sound velocity, the profile of the absorption per kilometer β, and the profile of the density of the medium in a layered waveguide of thickness H = 10.24 km. Around a depth of 5.2 km there is a smooth density-transition layer with a thickness of 0.1 km. The solid curve in Fig. 2 represents the distance profile of the intensity of the sound field, calculated according to (16) by the numerical scheme of fractional steps with Fourier transform,[1] for a point source situated at a depth of 0.15 km and operating at a frequency of 50 Hz. The dots represent the intenstiy of the field of the same source, calculated by the summation of normal modes on the basis of the algorithm in Ref. 8 for a model of the ocean where, rather than a layer, a homogeneous half-space is situated below a depth of 5.2 km and the density varies discontinuously at this depth. Both profiles are plotted with distance at the level of the source. The agreement of the results is deemed satisfactory.

Thus, the introduction of operator notation makes it possible to explain the physical significance of the parabolic equation method and to formulate equations (10) and (16) which, in contrast with the Leontovich–Fok equation, contain an additional term describing the amplitude variation of an approximately plane wave. The analogy with the one-dimensional WKB method can be exploited to deduce the initial conditions (15) for the field of a divergent wave.

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Influence of the angle of taper of the sonic nozzle on the operation of a stem-jet generator

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The results of experimental studies of the frequency, acoustic power, and efficiency of a stem-jet sound generator as a function of the angle of taper of the nozzle are reported. Certain gasdynamic characteristics of jets from nozzles with different angles and the same differential pressure are determined simultaneously.

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It is well known that changing the angle of taper of convergent (sonic) nozzles operating at supercritical differential pressures not only affects the velocity diagram and mass-flow characteristics of the gas flow, but also changes the outer contours of the underexpanded jet "cells" and the configuration of the sonic line due to the advent of radial velocity components.[1] It has been observed, moreover, that with a variation of the taper angle (denoted by 2α) the "cell" dimensions also change; for example, the length of the first cell of a cylindrical jet has a maximum near $\alpha \simeq 50°$ (Ref. 2). Inasmuch as the cell structure and shape determine the longitudinal gradient of the static pressure in the jet as well as the position of the compression shock, a variation of the nozzle taper will certainly also affect the operation of gas-jet sound generators, which utilize the instability effect of a supersonic jet when it is impeded by an obstruction (Hartmann effect). However, the literature does not contain any data on the effects of this factor on sound generation, and the usual tendency for increasing the radiated power is to increase the kinetic energy of the initial jet by equipping the generators with nozzles having the maximum discharge coefficient ($\mu = 1$), i.e., profiled ($\alpha \simeq 0°$) or slightly convergent nozzles.

Here we give the results of measurements designed to explain the influence of the nozzle taper angle on the acoustical parameters of the generator. Certain gasdynamic characteristics of the jet have been measured simultaneously in order to determine the causes of the observed influence.

The investigations were carried out on a stem-type gas-jet generator,[3] in which, rather than a cylindrical jet as in the Hartmann generator, an annular jet is used, which moves along a circular cylinder (stem) set up on the nozzle axis. Our previous measurements of the cell length of such a jet over a wide range of differential pressures and nozzle-stem diameter ratios[4] have shown that the cell length does not have extremal values; it decreases approximately linearly as α is increased, although for small differential pressures $P_0/P_\infty < 4.5$ (where P_0 is the pressure in the prechamber and P_∞ is the pressure in the surrounding medium), a small width of the annular slot δ (see Fig. 1), and angles $\alpha < 10°$, under the influence of the boundary layer created in the nozzle, this relationship can be distorted somewhat. In this study we carried out the measurements on calibrated nozzles ($d_c = 6$ mm, $\delta = 1$ mm) with variation of the angle α from 0 to 90° in 15° increments. For the system with $\alpha = 0°$ we used a Vitoshinskii nozzle with a short cylindrical section at the exit. To preclude the possibility of variation of the acoustic radiation conditions associated with the influence of nearby reflecting surfaces, all the nozzles had the same overall dimensions, including the exterior angle $\beta = 90°$. All the measurements were carried out for a fixed value of the compressed-air pressure: $P_0/P_\infty = 3.9$.

The gasdynamic measurements performed in the jet without the resonator entailed an investigation of the jet structure by the optical shadow method and a determination of the longitudinal (x direction in Fig. 1) distribution of the static pressure along the surface of the stem. The cell length Δ, which is determined by the arrival of the first oblique compression shock at the surface of the jet, and the positions of the oblique shocks (see Fig. 1) were evaluated from the photographs within ±5% error limits. The static pressure was measured through three drain ports located around the circumference of the stem and communicating through the central duct with a MDD-0-1600 diaphragm pressure gauge. The stem itself was capable of moving in the bushings along the axial direction; its displacement was monitored with a linear potentiometer. The voltages taken from the potentiometers of the pressure and displacement gauges were transmitted to the inputs of an x−y recorder. During motion of the stem the

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