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The calculation of the harmonic sound fields in the waveguides by the corrected wide-angle parabolic approximation

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Consider the calculation of the sound field of the harmonic point source in the 2D liquid waveguide with the depth H, the complex wavenumber K and the density R all of them depending on both coordinate and with the parallel boundaries.

$$R(x,z)\left(\frac{\partial}{\partial x}\frac{1}{R(x,z)}\frac{\partial p}{\partial x}+\frac{\partial}{\partial z}\frac{1}{R(x,z)}\frac{\partial p}{\partial z}\right)+K^{2}(x,z)\,p=\delta(0)\,\delta(z-z_{s})$$

$$p(x,0)=p(x,H)=0,\quad \mathrm{Im}\,K>0\ \Rightarrow\ \lim_{x^{2}+z^{2}\to\infty}p(x)=0 \qquad \text{(I)}$$

Now we rewrite the boundary problem (1) in the operator notation

Here

$$\vec{p}(x)=p(x,z),\quad \hat{R}(x),\ \hat{T}(x)\quad \begin{pmatrix}0 & \hat{R}(x)\\ \hat{T}(x) & 0\end{pmatrix}\begin{pmatrix}\vec{p}\\ \dfrac{\partial\vec{p}}{\partial x}\end{pmatrix} \qquad (2)$$

$$K(x)\,|\qquad \|\frac{\partial p}{}\ |=|$$

$$\vec{p}(x)=p(x,z),\quad \hat{R}(x),\ \hat{T}(x)$$

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are operators acting as

$$\hat{R}(x)\,\vec{p}(x)=R(x,z)\,p(x,z)$$

$$\hat{T}(x)\,\vec{p}(x)=R(x,z)\frac{\partial}{\partial z}\frac{1}{R(x,z)}\frac{\partial p(x,z)}{\partial z}+K^{2}(x,z)\,p(x,z)$$

with the domain of definition consisting of functions

$$\vec{p}(x)$$

meeting the conditions on the waveguide boundaries.

Let us now seek the solution for the homogeneous equation corresponding to the problem (2) as the sum of waves propagating in both directions. To this end we use the einsatz for the independent variables

$$\begin{pmatrix}\vec{p}\\ \dfrac{\partial\vec{p}}{\partial x}\end{pmatrix}=\begin{pmatrix}1 & 1\\ i\sqrt{\hat{T}(x)} & -i\sqrt{\hat{T}(x)}\end{pmatrix}\begin{pmatrix}\vec{p}_{+}\\ \vec{p}_{-}\end{pmatrix}$$

After all necessary transformations we get

$$\frac{\partial}{\partial x}\begin{pmatrix}\vec{p}_+\\ \vec{p}_-\end{pmatrix} = \Bigg[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} i\sqrt{\hat{T}(x)} - \qquad (3)$$

$$-\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\frac{1}{\sqrt{\hat{T}(x)}}\left(\frac{\partial\sqrt{\hat{T}(x)}}{\partial x} - \frac{\partial\hat{R}(x)}{\partial x}\frac{1}{\hat{R}(x)}\sqrt{\hat{T}(x)}\right)\Bigg]\begin{pmatrix}\vec{p}_+\\ \vec{p}_-\end{pmatrix}$$

From (3) one can see that the separation of waves of two directions is exact only in the layered waveguide. In the range-dependent waveguide the order in the norm of operator T of the nondiagonal terms that describe the interaction of waves of two directions are smaller than the diagonal ones. Assuming that R and T depend on x slowly we take in (3) only diagonal terms neglecting the nondiagonal ones to get the independent equations for waves of both directions, for example, for the wave propagating from left to right:

$$\frac{\partial\vec{p}_+}{\partial x} = i\sqrt{\hat{T}(x)}\,\vec{p}_+ \qquad (4)$$

This is the operator analog for the main WKB approximation. From the other point of view this is the operator notation for the adiabatic normal mode method complemented by the taking into account the interaction of normal modes propagating in one direction. The initial condition for the wave originating from the source results from the comparison with the solution for the layered waveguide

$$\vec{p}_+(0) = \left(2\sqrt{\hat{T}(0)}\right)^{-1}\vec{\delta}$$

We can solve the equation (4) using the simple predictor

$$\vec{p}_+(x+h) = \exp\left[ih\sqrt{\hat{T}(x)}\right]\vec{p}_+(x) \qquad (5)$$

The parabolic equation technique by Leontovich-Fok follows from (5) if we use for the step operator in (5)

$$\exp\left[ih\sqrt{\hat{T}(x)}\right]$$

the simple Tailor expansion (here R=1)

$$\exp\left[ih\sqrt{\hat{T}(x)}\right] \approx \exp\left[ih\left(K_0 + \frac{1}{2K_0}\frac{\partial^2}{\partial z^2} + \frac{K^2(x,z)-K_0^2}{2K_0}\right)\right] \qquad (6)$$

Use of that expansion assumes the small values of the transversal wavenumber of the sound field under consideration and the small difference of the sound speed profile from the constant one.

The Hardin-Tappert technique uses the smallness of the commutator

$$\left[\partial^2/\partial z^2 \,,\; K^2(x,z)\right]$$

To solve (6) according to

$$\exp\left[ih\left(K_0 + \frac{1}{2K_0}\frac{\partial^2}{\partial z^2} + \frac{K^2(x,z) - K_0^2}{2K_0}\right)\right] =$$

$$= \exp[ihK_0]\exp\left[ih\frac{K^2(x,z)-K_0^2}{4K_0}\right]\exp\left[\frac{ih}{2K_0}\frac{\partial^2}{\partial z^2}\right]\exp\left[ih\frac{K^2(x,z)-K_0^2}{4K_0}\right] + O(h^3)$$

We propose to compute the step operator using the rational Pade approximations for the function

$$\exp[ih\sqrt{\lambda}]$$

of the complex variable λ. Particularly, one can use the known Pade approximation for the function

$$\exp[ih\varkappa]$$

and

$$\varkappa = \sqrt{\lambda}\ [5]$$

to produce the rational approximation in the whole complex plane delivering any given precision in the given vicinity of zero and infinity

$$\exp\left[ih\sqrt{\lambda}\right] = \prod_{\ell=1}^{N}\frac{\lambda - \mu_\ell''(h)}{\lambda - \mu_\ell(h)} + E_N,\quad \lim_{N\to\infty} E_N = 0 \qquad (7)$$

Here “ denotes the complex conjugate. All

$$\mu_\ell$$

Turn out to lie in this case in the lower halfplane. In the computations we always use the finite-dimensional approximations to differential operators, namely, the matrices. The spectra of matrices always lie in the bounded part of the complex plane λ. Using the definition of the function with operator argument [6] we generalize (7) for the matrix argument

$$\exp\left[ih\sqrt{\hat{T}}\right] = \frac{1}{2\pi i}\int_\Gamma \exp[ih\sqrt{\lambda}]\,(\hat{T}-\lambda)^{-1}\,d\lambda \approx$$

$$\approx \frac{1}{2\pi i}\int_\Gamma \prod_{\ell=1}^{N}\frac{\lambda - \mu_\ell''(h)}{\lambda - \mu_\ell(h)}\,(\hat{T}-\lambda)^{-1}\,d\lambda = \prod_{\ell=1}^{N}\frac{\hat{T} - \mu_\ell''(h)}{\hat{T} - \mu_\ell(h)} \qquad (8)$$

Here the path Γ encircles the spectrum of the operator T letting it left. Due to the validity of the inequality

$$\operatorname{Im} K \geq 0$$

the action by the operators (8) is absolutely stable. The numerical implementation of the step operator (8) consists only from matrix multiplication and solution of the system of linear algebraic equations. We used the integral equalities by Marchuk [7] to approximate the differential operator T. The Figure shows the results of calculation for the layered waveguide to be in good agreement with the normal mode technique results.

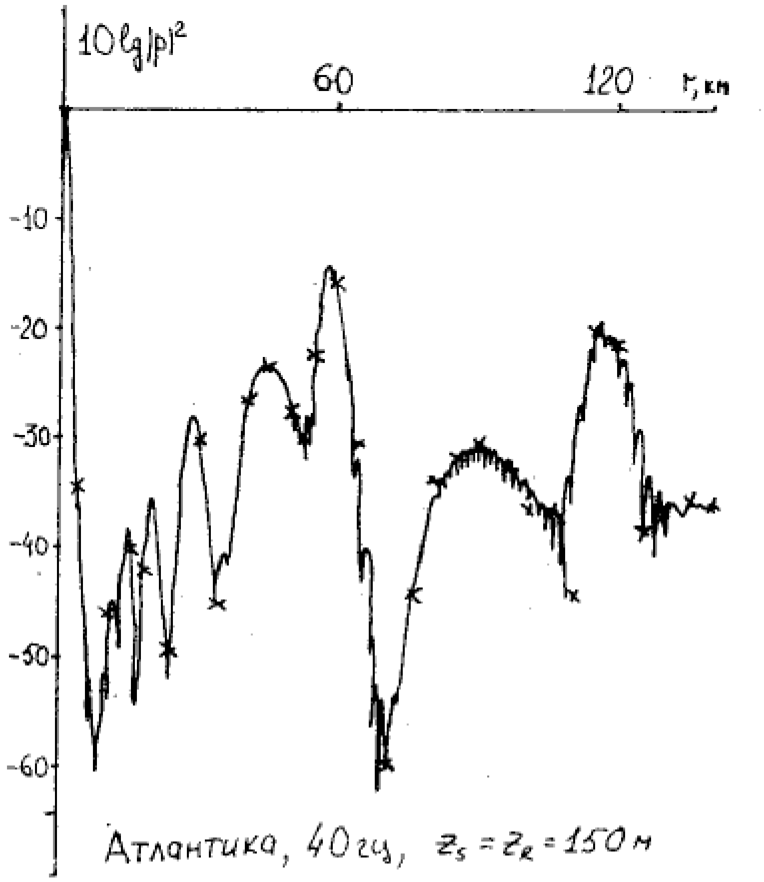

Fig. Atlantic Ocean, 40 Hz, source and receiver depths are of 150 m.

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