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AN EFFECTIVE NUMERICAL SOLUTION OF GUIDED WAVES EQUATIONS

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Resume – Le problème aux limites pour un guide d'ondes bidimensionnel scalaire dont la vitesse de son et la densité dependent lentement de x et arbitrairement de z est resolu par la méthode de factorization additive. L'équivalence entre cette méthode et la solution monodirectionnelle aux modes couplés est prouvéeÿ L'approprié problème Cauchy pseudodifferentiel est resolu avec la précision donnée par un algorithme fondé sur l'approximation Padé de grand ordre. Le rendement d'une réalization programme, adaptée pour les calculations de son sous-marin est discuté.

Abstract – The boundary problem for the two-dimensional scalar acoustic waveguide with sound speed and density slowly varying with x and arbitrary on z is treated by the method of additive factorization. The equivalence between such a method and the coupled modes one-way solution is proved. The resulting initial pseudodifferential problem is numerically solved with any given accuracy by the high order Pade-type approximation algorithm. The performance of the computer implementation, adopted to underwater sound propagation calculations is discussed.

I – INTRODUCTION

Many problems of waves propagation such as underwater sound propagation, long range radio waves propagation, marine seismology, light fiber propagation are of interest in case not only of layered media, but of media with properties varying in two directions. Among the mathematical procedures to predict waves fields in such circumstances the most used are the method of coupled modes /1,2/ and the parabolic equation technique in its manifold realizations /3,4/. In the relation following the equivalence of coupled modes or guided waves equations to the appropriate abstract parabolic equation is established and the algorithm to solve the simplified form of the last is proposed, featuring the low computation cost at the any given accuracy.

II. – GUIDED WAVES EQUATIONS OPERATOR FORM.

As a model of acoustical waves propagation process we shall use the system of differential equations of type

$$\left|\begin{array}{ccc} D_x & 0 & -i\omega\rho \\ D_z & -i\omega\rho & 0 \\ -i\omega\rho^{-1}C^{-2} & D_z & D_x \end{array}\right| \left|\begin{array}{c} p \\ v_z \\ v_x \end{array}\right| = \left|\begin{array}{c} f_x \\ f_z \\ V \end{array}\right| , \qquad (x,z)\in(-\infty,\infty)\times(0,H), \qquad (1)$$

$$Y_0 p(x,0)+v_z(x,0)=0,\quad Y_H p(x,H)+v_z(x,H)=0$$

p being acoustical pressure, v_x,v_z-acoustical velocity cartesian components, f_x,f_z – cartesian components of external forces density, V-external volume velocity density, all of them being functions of cartesian horizontal coordinate x or range and z –

vertical coordinate or depth. D_x and D_z are partial derivatives on x and z, ω-cyclic frequency, ρ-medium density, C-medium sound speed, $i^2=-1$, Y_0 and Y_H-acoustical admittances at upper and lower boundaries of waveguide, H – the maximum depth of the waveguide, taken into account. First two equations of (1) are the Newton equations, the third is the continuity equation. Deleting from (1) the vertical component of the acoustical velocity v_z, we obtain the next operator form of (1):

$$\begin{vmatrix} D_x p \\ D_x v \end{vmatrix} = \begin{vmatrix} 0 & i\omega\mathbb{R} \\ i\omega\mathbb{B} & 0 \end{vmatrix} \begin{vmatrix} p \\ v \end{vmatrix} + \begin{vmatrix} f_x \\ W \end{vmatrix}, \begin{matrix} p(x)=p(x,z),\ v_x(x)=v(x,z),\ \mathbb{R}v=\rho(x,z)v_x(x,z), \\ f_x(x)=f_x(x,z),\ W(x)=V(x,z)-D_z(i\omega\rho(x,z))^{-1}f_z(x,z) \end{matrix} \tag{2}$$

$$\mathbb{B}p(x)=(\hat{\rho}^{-1}(x,z)C^{-2}(x,z)-D_z((i\omega)^2\rho(x,z))^{-1}D_z)p(x,z),$$

$$Y_0 p(x,0)+(i\omega\rho(x,0))^{-1}D_z p(x,0)=0,\quad Y_H(p(x,H)+(i\omega\rho(x,H))^{-1}D_z p(x,H)=0$$

Equations (2) can be seen as operator form of guided waves equations /1,2/: assuming for the sake of simplicity $\mathbb{R}=\mathbb{I}$-identity operator and taking into account that the operator $\mathbb{B}$ in the local normal modes $\phi_l(x,z)$ basis has the diagonal form

$$\mathbb{B}(x) = \hat{\Phi}(x)\ \mathbb{C}^{-2}(x)\ \hat{\Phi}^{-1}(x),\quad \hat{\Phi}(x) = \underset{l}{\mathrm{row}}\{\vec{\phi}_l(x)\},\ \mathbb{C}(x) = \underset{l}{\mathrm{diag}}\{C_l\}$$

C_l being the phase velocity of l-th local normal mode, we obtain for the local normal modes amplitudes $\vec{a}(x)=\hat{\Phi}^{-1}(x)p(x)$, $\vec{b}(x)=\hat{\Phi}^{-1}(x)v(x)$:

$$D_x a_l(x) - i\omega b_l(x) = -\sum_k \gamma_{lk}(x)a_k(x),\quad D_x b_l(x) - i\omega C_l^{-2}(x)a_l(x) = -\sum_k \gamma_{lk}(x)b_l(x),$$

where $\gamma_{lk}=\{\hat{\Phi}^{-1}(x)(D_x\hat{\Phi}(x))\}_{lk}$ are the local normal modes coupling coefficients.

III. ADDITIVE FACTORIZATION AND ONE-WAY SOLUTIONS.

In the subsequent discussion we shall use functions f of linear operators $\mathbb{T}$ accordingly to the Riss's definition:

$$f(\mathbb{T}) = (2\pi i)^{-1}\int_\Gamma f(\lambda)(\mathbb{T}-\lambda)^{-1}d\lambda \tag{3}$$

where the path of integration Γ in the complex plain is encircling the spectre of operator $\mathbb{T}$, letting him left and the operator $\mathbb{T}$ supposed to be such, that the integral does converge.

We shall now reformulate equations (2) in the manner of Wentzel-Kramer-Brillouin method. Let us assume, that outside of interval $x\in(a,b)$ our waveguide is layéred and all sources are situated inside this interval. Introducing the new local amplitudes of waves of two opposite directions:

$$\begin{vmatrix} p \\ v \end{vmatrix} = \begin{vmatrix} \mathbb{Y}^{-1}(x) & \mathbb{Y}^{-1}(x) \\ \mathbb{Z}(x) & -\mathbb{Z}(x) \end{vmatrix} \begin{vmatrix} u^+ \\ u^- \end{vmatrix},\ \begin{matrix} \mathbb{S}(x)=\sqrt{\sqrt{\mathbb{R}(x)}\ \mathbb{B}(x)\ \sqrt{\mathbb{R}(x)}}, \\ \mathbb{Y}(x)=\sqrt{\mathbb{S}(x)}\ \sqrt{\mathbb{R}(x)}^{-1};\ \mathbb{Z}(x)=\sqrt{\mathbb{R}(x)}^{-1}\sqrt{\mathbb{S}(x)} \end{matrix}$$

and taking into account that outside of the interval (a,b) only outgoing waves must exist, we obtain by this process of additive factorization / 6 / the following system of equations:

$$\begin{vmatrix} \mathbb{P} & \mathbb{U} \\ \mathbb{L} & \mathbb{Q} \end{vmatrix} \begin{vmatrix} u^+ \\ u^- \end{vmatrix} = \begin{vmatrix} F^+ \\ F^- \end{vmatrix},\quad \begin{vmatrix} F^+ \\ F^- \end{vmatrix} = 1/2 \begin{vmatrix} \mathbb{Y} & \mathbb{Z}^{-1} \\ \mathbb{Y} & -\mathbb{Z}^{-1} \end{vmatrix} \begin{vmatrix} f_x \\ W \end{vmatrix}$$

$$\mathbb{P} = D_x - i\omega\mathbb{S}(x) + 1/2\ (\ \mathbb{Z}^{-1}(x)(D_x\mathbb{Z}(x)) - \mathbb{Y}^{-1}(x)(D_x\mathbb{Y}(x))\ ),\quad u^+(a)=0$$

$$\mathbb{Q} = D_x + i\omega\mathbb{S}(x) + 1/2\ (\ \mathbb{Z}^{-1}(x)(D_x\mathbb{Z}(x)) - \mathbb{Y}^{-1}(x)(D_x\mathbb{Y}(x))\ ),\quad u^-(b)=0$$

$$\mathbb{L} = \mathbb{U} = -\ 1/2\ (\ \mathbb{Z}^{-1}(x)(D_x\mathbb{Z}(x)) + \mathbb{Y}^{-1}(x)(D_x\mathbb{Y}(x))\ )$$

witch can be solved by the Gauss-Zeidel iterations:

$$u_k^+ = \mathbb{P}^{-1}(-\mathbb{U}u_{k-1}^- + F^+),\ u_k^- = \mathbb{Q}^{-1}(-\mathbb{L}u_k^+ + F^-) \tag{4}$$

converging, if $\| \mathbb{P}^{-1}\mathbb{U}\mathbb{Q}^{-1}\mathbb{L}\| < 1$ for some norm $\|\cdot\|$. If ρ and β are slowly varying with x, then $\mathbb{U}$ and $\mathbb{L}$ are small and for well defined Cauchy's operators $\mathbb{P}$ and $\mathbb{Q}$ the convergence of (4) is very probable. Assuming now zero field as zero order approximation for the first order approximation of one-way positive direction wave u_1^+ we obtain $\hat{\mathbb{P}}u_1^+ = F^+$, yielding in the local normal modes basis the one-way guided waves equations /5/:

$$D_x c_l - i\omega C_l^{-1} c_l = -\sum_k \gamma_{lk}(C_l + C_k)(2C_lC_k)^{-1} + \{\hat{\Phi}^{-1}F^+\}_l$$

Supposing the coupling coefficients γ_{lk} to be fast diminishing with $|l-k|$ and therefore letting $(C_l+C_k)/(2C_lC_k)^{-1} \cong 1$ neglecting so the difference of transmission from identity, we can obtain then in operator form the abstract parabolic equation

$$D_x u_1^+ - i\omega\mathbb{S}(x)u_+^1 = F^+ \tag{5}$$

giving rise to the family of known PEs, according to the technique employed to approximate $\mathbb{S}$ and to solve the resulting approximated equation.

IV. NUMERICAL SOLUTION.

To solve (5) on the x grid with step h it is convenient to use the algorithm $u^+(x+h) = \exp(i\omega h\mathbb{S}(x))u^+(x)$ of exponential fitting type /3,6,7/. To approximate the propagation factor operator $\mathbb{E}=\exp(i\omega h\mathbb{S}(x))$ with any given accuracy we use the rational approximation: if $f(\lambda)\cong F_n(\lambda)/G_m(\lambda)$ (F_n, G_m-polynomials of degrees n,m) in the vicinity of the operator's $\mathbb{S}$ spectre then (3) gives $f(\mathbb{S})\cong F_n(\mathbb{S})(G_m(\mathbb{S}))^{-1}$. We construct the appropriate approximation to $\exp(ih\sqrt{\lambda})$ with the known Pade approximations /5/:

$$\exp(\lambda) = \left[M_n(\lambda^2)+\lambda N_n(\lambda^2)\right]\left[M_n(\lambda^2)-\lambda N_n(\lambda^2)\right]^{-1}+E_n(\lambda), \quad \sqrt{\lambda} = F_m(\lambda)(G_m(\lambda))^{-1}+H_n(\lambda)$$

M_n, N_n, F_m, G_m being polynomials with real coefficients, computed by recurrence, E_m, H_n- approximation errors known to fade with increasing m and n in the complex plain without the negative half of real axes. Combining this approximations we get

$$\begin{aligned}\exp(ih\sqrt{\lambda}) &= \left[G_m(\lambda)M_n(-h^2\lambda)+ihF_m(\lambda)N_n(-h^2\lambda)\right]\left[G_m(\lambda)M_n(-h^2\lambda)-ihF_m(\lambda)N_n(-h^2\lambda)\right]^{-1} = \\ &= \prod_{k=1}^{k=n+m}\left[\lambda - \mu_k(h)\right]\left[\lambda - \mu_k^*(h)\right]^{-1}\end{aligned} \tag{6}$$

μ_k being the roots of nominator in the above fraction, μ_k^* - their complex conjugated We have computed μ_k for some m,n,h. All of them are situated in the IV quadrant of complex plain far enough from the spectre of $\mathbb{S}$, lying in the I quadrant, the fact leading to stability of action by $\mathbb{E}$. The error of (6) is decreasing with n, m increasing, but practically important is to approximate the region of spectre, corresponding to propagating local normal modes, lying in the vicinity of (1,0) in the upper halfplain. For example, $m=4, n=4, h=4\pi$ give for local normal modes with Brillouin angles up to 60° the phase error less than $5*10^{-6}$ rad enabling underwater sound propagation calculations to the range of 9000km with frequencies up to 100Hz with absolute phase error less then $\pi/2$.

The computer implementation of the above technique needs a discretization of vertical coordinate z and an appropriate discrete approximation of $\mathbb{S}$. This can be done by finite-differences techniques, Galerkin method, Marchuk's equalities method and so on /6/. The common feature of such techniques is that the operator $\mathbb{B}$ can be approximated by the product $\hat{B}^{-1}\hat{A}$ of band matrices $\hat{B}$ and $\hat{A}$, while $\mathbb{R}$ is approximated by the diagonal matrix $\hat{R}$, giving for $\mathbb{E}$:

$$\begin{aligned}\hat{E} &\cong \prod_k (\sqrt{\hat{R}}\,\hat{B}^{-1}\hat{A}\sqrt{\hat{R}} - \mu_k)(\sqrt{\hat{R}}\,\hat{B}^{-1}\hat{A}\sqrt{\hat{R}} - \mu_k^*)^{-1} = \\ &= \prod_k \left[\hat{I} - 2\mathrm{Im}(\mu_k)(\hat{A}\sqrt{\hat{R}} - \mu_k\hat{B}\sqrt{\hat{R}}^{-1})^{-1}\hat{B}\sqrt{\hat{R}}^{-1}\right]\end{aligned} \tag{7}$$

- an easy to implement algorithm, including multiplication by band matrices and solving systems of equations with such matrices. Taking now into account, that the rational approximation to $\mathbb{E}$ is fully determined by the set of μ_k, we can include into our consideration also the case of admittances Y_0, Y_H depending on local normal mode phase velocity, and henceforth, on spectral parameter μ_k, as for underlying semispace with known properties. This assumption leads to the dependence of $\hat{A}$ and $\hat{B}$ on μ_k due to their dependence from boundary conditions in $\mathbb{B}$ while the above form of rational approximation to $\mathbb{E}$ remains unchanged. This feature is unique to the propagation factor algorithm (6,7). The another advantage of (6,7) is the aggregate approximation of equation (5) and his solution resulting in lower calculations cost comparably to other known techniques.

The three-dimensional cylindrically-symmetric problem may be treated in a similar way, substituting the propagation factors exp(+-iωsx) by zero order Hankel functions $H_0^{1,2}(\omega sr)$. Use of simplest asymptotic expansion $(2/i\pi\omega sr)^{1/2}\exp(i\omega sr)$ results then in the same propagation factor algorithm (6,7).

The package of FORTRAN-coded routines is developed to calculate the long-range underwater sound propagation in range-depth dependent environment. Because of main interest given in this case to the sound pressure values, it uses a slightly different factorization:

$$\begin{vmatrix} p \\ v \end{vmatrix} = \begin{vmatrix} \mathbb{V}(x) & \mathbb{V}(x) \\ \mathbb{W}(x) & -\mathbb{W}(x) \end{vmatrix} \begin{vmatrix} p^- \\ p^+ \end{vmatrix}, \quad \begin{matrix} \mathbb{V}(x) = \sqrt{\mathbb{R}(x)}\,, \\ \mathbb{W}(x) = \sqrt{\mathbb{R}(x)}^{\,-1}\mathbb{S}(x) \end{matrix}$$

with propagation factor algorithm also remained unchanged. The performance of this package may be approximately evaluated as 50 floating points operations per one wavelength on x per one node in grid on z having the typical value of one fourth of the wavelength by sufficient accuracy for Brillouin angles up to 35°.

V. – CONCLUSION.

We have used operator notation of fluid acoustic's equations to get the pseudo-differential parabolic equation, governing the one-way propagation of harmonic sound in range-depth dependent environment and proposed an effective numerical algorithm to solve it. In the following relations we shall include into consideration elastic media, fields with arbitrary time dependence and formulate a three-dimensional technique for horizontally slowly varying waveguides.

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